Coincidence Problem and Combination Resonance

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The similarity between coincidence problem in jet engines and combination resonance in the classical theory of parametric excitation is pointed out. The results corresponding to the deterministic excitation are generalized to include the stationary narrow-band Gaussian random excitation, and in this manner the similarity between the combination resonance caused by harmonic excitations and that induced by narrow-band random excitations, is illustrated. Finally, simple stability criteria are given in power series of a small parameter ϵ, and the first-order terms are explicitly calculated.

I. Introduction

THE high-pressure compressor of a jet engine includes labyrinth air seal systems (see Fig. 1). In the past some of these systems have failed due to the so-called "coincidence" problem. Mathematically this problem can be stated in the following way. First, let ω_R be the natural frequency of the rotating seal; ω_R is a function of the number of nodal diameters k and the natural frequency of the static seal ω_s, which itself is a function of the number of nodal diameters k. Then, if θ is the rotating speed of the rotor, the coincidence occurs for the condition‡

$$\omega_R - \omega_s = k\theta \qquad k=2,3,... \tag{1}$$

Today it is a standard practice in the development of both military and commercial jet engines to avoid this coincidence condition in the operational range, as one of the design criteria.

Consider now the classical theory of parametric excitation. It is well known that the so-called combination resonance occurs if the following condition is satisfied[2]

$$|\omega_\alpha \pm \omega_\beta| = k\theta \qquad k=1,2,... \tag{2}$$

where ω_α and ω_β, $\alpha \neq \beta$, $\alpha,\beta = 1,..., n$, are the natural frequencies of the system, and θ is the excitation frequency. Comparing Eqs. (1) and (2), we immediately recognize the similarity between coincidence problem in jet engines and combination resonance in the classical theory of parametric excitation. Remarkably, we also know that the combination resonance of the first kind, i.e.,

$$\omega_\alpha + \omega_\beta = k\theta$$

may not occur if the excitation force is nonconservative; see Refs. 2-6. For this class of excitation, only the second kind; i.e., $|\omega_\alpha - \omega_\beta| = k\theta$, $\alpha \neq \beta$, $\alpha,\beta = 1,...,n$; may take place. As is known, aerodynamic forces are of nonconservative nature. Hence, in the actual test, we have never observed the coincidence of the kind $\omega_R + \omega_s = k\theta$ in jet engines. This remarkable similarity between these two problems is interesting enough to be further explored. The purposes of this paper are to 1) point out the similarity between these two problems in an effort to bring two groups—the practicing aerospace engineers and academic researchers—together so that each group can benefit from the other's experience; 2) generalize the results corresponding to the deterministic excitation to include the stationary narrow band Gaussian random excitation in order to illustrate the similarity between combination resonances caused by harmonic excitations and those induced by narrow-band random excitations; and 3) provide simple stability criteria. These criteria appeared in the first author's dissertation,[7] where general systems with n degrees of freedom have been discussed. For a two-degree-of-freedom system, similar results have been obtained by Wedig in Germany.[8] Also, Ariaratnam[9] has reported similar results for a general system with n degrees of freedom. The results reported here are slightly more conservative than Ariaratnam's results, but up to the first order of magnitude in a small parameter ϵ, the differences are negligible. Since our criteria are simpler, we believe that the results are worthy of consideration.

In Sec. II, a mathematical model for structures subjected to stochastic and harmonic excitations is stated. In Sec. III, the basic results for a narrow-band Gaussian random excitation are summarized and compared with those of the classical theory of harmonic excitation. A brief discussion relating to the analysis of random excitations is presented in Sec. IV, and in Sec. V a simple example is given for illustration. Two appendices are attached to this paper. In Appendix A, we compare our stability conditions with those given by Ariaratnam,[9] and in Appendix B we prove a subsidiary result which is used in Sec. III.

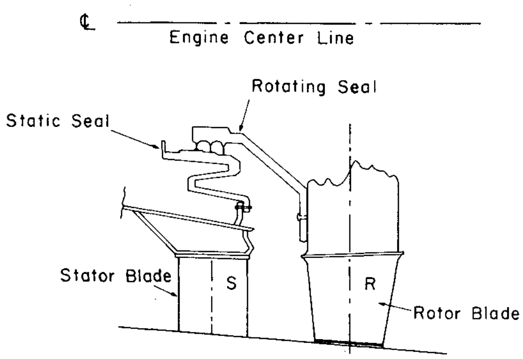

Fig. 1 Labyrinth air seal systems.

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‡In practice this coincidence is avoided as a design condition. Traditionally, coincidence problem has been investigated from the point of view of the traveling-wave vibration.[1]

II. Mathematical Model

Through either finite element technique[10] or other methods,[11] many dynamic problems of structural systems can be described by the following discretized equations

$$\ddot{x}_\alpha + \epsilon \sum_{\beta=1}^{n} C_{\alpha\beta}\dot{x}_\beta + \sum_{\beta=1}^{n} [\delta_{\alpha\beta}\omega_\beta^2 + \epsilon P_{\alpha\beta} r(t)] x_\beta = 0$$
$$\alpha = 1,\ldots,n \tag{3}$$

where $\boldsymbol{x} = \{x_\alpha\}$ is a set of generalized coordinates; $\boldsymbol{C} = [C_{\alpha\beta}]$ is a time-independent matrix with positive diagonal elements; $\boldsymbol{P} = [P_{\alpha\beta}]$ is a time-independent matrix; $\delta_{\alpha\beta}$ is the Kronecker delta; ω_β, $\beta = 1,\ldots,n$, are the natural frequencies of the structure, which are assumed to be distinct, i.e., $\omega_\alpha \neq \omega_\beta$ if $\alpha \neq \beta$ for all α and β; ϵ, $(0 < \epsilon \ll 1)$, is a small parameter which indicates that damping and the time variations of the excitation forces are small in magnitude; and $r(t)$ is either a harmonic function [i.e., $r(t) = \cos\theta t$] or a stationary Gaussian random process with zero mean.

For simplicity, we assume that the initial disturbances, i.e., the initial conditions for Eqs. (3), are deterministic. Under this consideration we use the following definition of the stochastic stability.

Definition

The trivial solution of Eqs. (3) is *stable in the mean-square sense* if, for a given $\nu > 0$, there exists a $\delta(\nu) > 0$ such that, for any prescribed initial condition $|x_\alpha(0)| < \delta$, we have $|E\{x_\alpha(t) x_\beta\}| < \nu$ for all $t \geq 0$ and $\alpha, \beta = 1,\ldots,n$, where $E\{\ldots\}$ denotes the expected value (or the ensemble average).

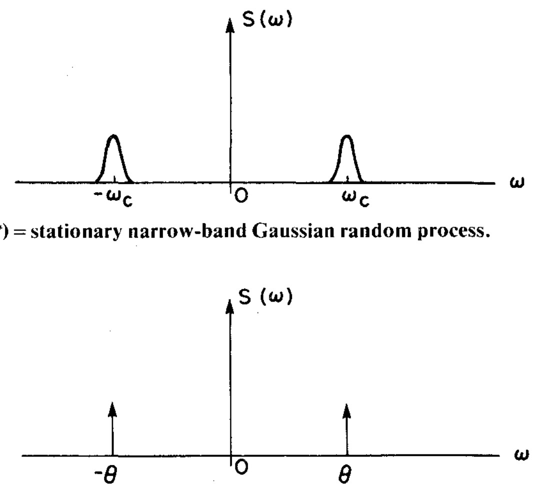
As is well known, the power spectrum of a stationary narrow-band Gaussian random process is very similar to that of a harmonic function[12] (see Fig. 2). Therefore, we expect that the frequency response and stability criteria for structures subjected to this kind of random loads have appearances similar to their counterparts in the case of harmonic excitations. In the next section, we present combination resonance, subharmonic resonance, and stability criteria for system (3), with $r(t)$ as either a stationary narrow-band Gaussian random process or a harmonic function.

III. Main Results

We discuss two separate cases (denoted by Secs. A and B, respectively) in this section.

A. $r(t)$ is a Stationary Narrow-Band Gaussian Random Process

In this case, we assume that the bandwidth $\Delta\omega$ of the significant portion of $r(t)$'s power spectrum, $S(\omega)$ is smaller



a) $r(t)$ = stationary narrow-band Gaussian random process.

b) $r(t) = \cos\theta t$.

Fig. 2 Power spectra of $r(t)$.

than the absolute value of the differences of any two natural frequencies of the structure, i.e.,

$$\Delta\omega < |\omega_\alpha - \omega_\beta| \qquad \alpha \neq \beta \qquad \alpha,\beta = 1,\ldots,n$$

Therefore, $S(\omega)$ is nonzero for $\omega_c - (\Delta\omega/2) \leq \omega \leq \omega_c + (\Delta\omega/2)$ and $-[\omega_c + (\Delta\omega/2)] \leq \omega \leq -[\omega_c - (\Delta\omega/2)]$ with ω_c denoting the central frequency, and $S(\omega)$ is almost zero outside of this range. For sufficiently small ϵ we then have the following combination resonance, subharmonic resonance, and mean-square stability conditions, which are, however, valid only to a first order of approximation in ϵ.

Combination Resonance of the First Kind

If the sum of any two natural frequencies ω_α and ω_β falls within the significant portion of the power spectrum $S(\omega)$, i.e., if we have

$$\omega_c - \frac{\Delta\omega}{2} \leq \omega_\alpha + \omega_\beta \leq \omega_c + \frac{\Delta\omega}{2} \qquad \alpha,\beta = 1,\ldots,n$$

then resonance occurs, and the condition under which the system can remain stable is§

$$C_{\alpha\alpha} + C_{\beta\beta} > \frac{\epsilon P_{\alpha\beta} P_{\beta\alpha}}{\omega_\alpha \omega_\beta} S(\omega_\alpha + \omega_\beta) \quad \text{for } P_{\alpha\beta}P_{\beta\alpha} \geq 0 \tag{4}$$

However, if $P_{\alpha\beta}P_{\beta\alpha} < 0$, then instability will not take place. The system will not have resonance problem. As we know, if the excitation force is conservative, the stiffness matrix is usually symmetric. Hence, under this kind of loading, $P_{\alpha\beta}P_{\beta\alpha}$ is positive, and the designer should be careful about the possibility of resonance of this type. On the other hand, if the excitation force is nonconservative, the stiffness matrix is usually nonsymmetric or, in many cases, antisymmetric. Hence, under this type of loading, $P_{\alpha\beta}P_{\beta\alpha}$ can be negative, and, if so, there will be no resonance problem of this kind. As we will see in the next case, nonconservative forces can create a different kind of combination resonance.

Combination Resonance of the Second Kind

If the difference of any two natural frequencies ω_α and ω_β falls within the significant portion of the power spectrum $S(\omega)$, i.e., if we have

$$\omega_c - \frac{\Delta\omega}{2} \leq |\omega_\alpha - \omega_\beta| \leq \omega_c + \frac{\Delta\omega}{2} \qquad \alpha \neq \beta$$

then resonance occurs, and the condition under which the system stays stable is

$$C_{\alpha\alpha} + C_{\beta\beta} > -\frac{\epsilon P_{\alpha\beta} P_{\beta\alpha}}{\omega_\alpha \omega_\beta} S(\omega_\alpha - \omega_\beta) \quad \text{for } P_{\alpha\beta}P_{\beta\alpha} \leq 0 \tag{5}$$

However, if $P_{\alpha\beta}P_{\beta\alpha} > 0$, then instability will not occur. The system will not have resonance problem. As we mentioned before that if the excitation force is nonconservative, the stiffness matrix is usually nonsymmetric or antisymmetric. Hence, under this type of loading, $P_{\alpha\beta}P_{\beta\alpha}$ can be negative, and the designer should be aware of the possibility of resonance of this kind.

Subharmonic Resonance

If twice of any natural frequency ω_α falls within the significant portion of the power spectrum $S(\omega)$, i.e., if we have

$$\omega_c - \frac{\Delta\omega}{2} \leq 2\omega_\alpha \leq \omega_c + \frac{\Delta\omega}{2}$$

§See Appendix A for comparison of this and the next two stability criteria with those given by Ariaratram.[9]

then resonance occurs, and the condition under which the system remains stable is

$$C_{\alpha\alpha} > \frac{\epsilon P_{\alpha\alpha}^2}{2\omega_\alpha^2} S(2\omega_\alpha) \tag{6}$$

B. $r(t) = \cos\theta t$

For the purpose of comparison, some of the classical theory of harmonic excitation is listed here. Detailed analysis of these results can be found in the original papers of Mettler,[13] Schmidt and Weidenhammer,[14] Valeev,[15] Hsu,[16,17] Weidenhammer,[18] and Fu and Nemat-Nasser.[4,19,20]

Combination Resonance of the First Kind

Similar to the case of the stationary narrow-band Gaussian random excitation, if the sum of any two natural frequencies ω_α and ω_β equals the excitation frequency θ, i.e., if we have

$$\omega_\alpha + \omega_\beta = \theta$$

then resonance occurs. The boundaries of unstable regions in the θ, ϵ-plane are defined by

$$\theta = \theta_0 \pm \frac{\epsilon}{2}\left[(C_{\alpha\alpha}/C_{\beta\beta})^{1/2} + (C_{\beta\beta}/C_{\alpha\alpha})^{1/2}\right]\left(\frac{P_{\alpha\beta}P_{\beta\alpha}}{4\omega_\alpha\omega_\beta} - C_{\alpha\alpha}C_{\beta\beta}\right)^{1/2} + 0(\epsilon^2)$$

where

$$\theta_0 = \omega_\alpha + \omega_\beta \qquad \frac{P_{\alpha\beta}P_{\beta\alpha}}{4\omega_\alpha\omega_\beta} - C_{\alpha\alpha}C_{\beta\beta} \geq 0 \text{ and } P_{\alpha\beta}P_{\beta\alpha} > 0$$

where the double sign (i.e., $\pm$) in the above equation refers to the two boundaries of each region of instability. If

$$C_{\alpha\alpha}C_{\beta\beta} > \frac{P_{\alpha\beta}P_{\beta\alpha}}{4\omega_\alpha\omega_\beta} \tag{7}$$

or $P_{\alpha\beta}P_{\beta\alpha} < 0$, then instability will not occur. The system will not have resonance problem. Condition (7) is comparable with Eq. (4) of the narrow-band random excitation. Furthermore, similar to the case of the narrow-band random excitation, the sign of $P_{\alpha\beta}P_{\beta\alpha}$ determines whether or not resonance occurs.

Combination Resonance of the Second Kind

If the difference of any two natural frequencies ω_α and ω_β equals the excitation frequency θ, i.e., if we have

$$|\omega_\alpha - \omega_\beta| = \theta$$

then resonance occurs. The boundaries of unstable regions in the θ, ϵ-plane are defined by

$$\theta = \theta_0 \pm \frac{\epsilon}{2}\left[(C_{\alpha\alpha}/C_{\beta\beta})^{1/2} + (C_{\beta\beta}/C_{\alpha\alpha})^{1/2}\right]\left(\frac{|P_{\alpha\beta}P_{\beta\alpha}|}{4\omega_\alpha\omega_\beta} - C_{\alpha\alpha}C_{\beta\beta}\right)^{1/2} + 0(\epsilon^2)$$

where

$$\theta_0 = |\omega_\alpha - \omega_\beta| \qquad \frac{|P_{\alpha\beta}P_{\beta\alpha}|}{4\omega_\alpha\omega_\beta} - C_{\alpha\alpha}C_{\beta\beta} \geq 0 \text{ and } P_{\alpha\beta}P_{\beta\alpha} < 0$$

If

$$C_{\alpha\alpha}C_{\beta\beta} > \frac{|P_{\alpha\beta}P_{\beta\alpha}|}{4\omega_\alpha\omega_\beta} \tag{8}$$

of $P_{\alpha\beta}P_{\beta\alpha} > 0$, then instability will not take place. There will be no resonance problem. Condition (8) is comparable with Eq. (5). Again, the sign of $P_{\alpha\beta}P_{\beta\alpha}$ decides whether or not the resonance of this type occurs.

Subharmonic Resonance

If twice of any natural frequency ω_α equals the excitation frequency θ, i.e., if we have

$$2\omega_\alpha = \theta$$

then resonance occurs. The boundaries of unstable regions in the θ, ϵ-plane are described by

$$\theta = \theta_0 \pm \epsilon\left(\frac{P_{\alpha\alpha}^2}{4\omega_\alpha^2} - C_{\alpha\alpha}^2\right)^{1/2} + 0(\epsilon^2)$$

where

$$\theta_0 = 2\omega_\alpha, \qquad \frac{P_{\alpha\alpha}^2}{4\omega_\alpha^2} \geq C_{\alpha\alpha}^2$$

If

$$C_{\alpha\alpha}^2 > \frac{P_{\alpha\alpha}^2}{4\omega_\alpha^2} \tag{9}$$

then instability will not happen. The system will not have resonance problem. Condition (9) is comparable with Eq. (6)

IV. Analysis

In this section, analysis for random excitation is given. To solve Eqs. (3) approximately by the method of the slowly varying phase and amplitude,[21] we first observe that when $\epsilon = 0$, we have

$$\ddot{x}_\alpha + \omega_\alpha^2 x_\alpha = 0 \qquad \alpha = 1,\ldots,n$$

which admits solutions of the form

$$x_\alpha = Q_\alpha \sin(\omega_\alpha t + \nu_\alpha)$$

where Q_α and ν_α are constants to be determined by the initial conditions. In accordance with the method of the slowly varying phase and amplitude, we then let Q_α and ν_α be slowly varying functions of time t, when $0 < \epsilon \ll 1$, and write

$$x_\alpha = Q_\alpha(t)\sin[\omega_\alpha t + \nu_\alpha] \tag{10}$$

$$\dot{x}_\alpha = Q_\alpha(t)\omega_\alpha\cos[\omega_\alpha t + \nu_\alpha] \tag{11}$$

Substituting Eqs. (10) and (11) into Eqs. (3), we arrive at

$$\frac{\dot{Q}_\alpha}{Q_\alpha} = -\frac{\epsilon}{\omega_\alpha}\varphi_\alpha(t)\cos[\omega_\alpha t + \nu_\alpha] \tag{12}$$

$$\dot{\nu}_\alpha = \frac{\epsilon}{\omega_\alpha}\varphi_\alpha(t)\sin[\omega_\alpha t + \nu_\alpha] \tag{13}$$

where

$$\varphi_\alpha(t) = \sum_{\beta=1}^{n}\left(\frac{Q_\beta}{Q_\alpha}\right)\Big[r(t)P_{\alpha\beta}\sin(\omega_\beta t + \nu_\beta) + C_{\alpha\beta}\omega_\beta\cos(\omega_\beta t + \nu_\beta)\Big]$$

Solving Eqs. (12) and (13), we obtain

$$Q_\alpha = Q_\alpha^0 e^{-\epsilon\eta_\alpha t} \qquad \nu_\alpha = \nu_\alpha^0 + \epsilon\xi_\alpha t$$

where Q_α^0 and ν_α^0 are constants to be determined by the initial conditions, and

$$\eta_\alpha = \frac{1}{t}\int_0^t \frac{\varphi_\alpha(\tau)}{\omega_\alpha}\cos(\omega_\alpha \tau + \nu_\alpha)\,\mathrm{d}\tau$$

$$\xi_\alpha = \frac{1}{t}\int_0^t \frac{\varphi_\alpha(\tau)}{\omega_\alpha}\sin(\omega_\alpha \tau + \nu_\alpha)\,\mathrm{d}\tau \tag{14}$$

Observing Eqs. (14), we immediately recognize that η_α and ξ_α are time averages of functions $(\varphi_\alpha/\omega_\alpha)\cos(\omega_\alpha\tau+\nu_\alpha)$ and $(\varphi_\alpha/\omega_\alpha)\sin(\omega_\alpha\tau+\nu_\alpha)$, respectively. Since ϵ is a positive quantity, for stability analysis we are only interested in the sign of the quantity

$$\lim_{t\to\infty}\eta_\alpha = \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\varphi_\alpha(\tau)}{\omega_\alpha}\cos(\omega_\alpha\tau+\nu_\alpha)\,\mathrm{d}\tau \tag{15}$$

In accordance with the method of the slowly varying phase and amplitude when we perform integration, we treat phase ν_α and amplitude Q_α [which are buried in φ_α in Eq. (15)], as constants. Keeping this in mind, we then are able to expand the right-hand side of Eq. (15) in the power series of ϵ as

$$\lim_{t\to\infty}\eta_\alpha(\epsilon) = \eta_{1\alpha} + \epsilon\eta_{2\alpha} + 0(\epsilon^2) \tag{16}$$

with

$$\eta_{1\alpha} = \lim_{t\to\infty}\eta_\alpha\Big|_{\epsilon=0} \qquad \eta_{2\alpha} = \lim_{t\to\infty}\frac{\partial\eta_\alpha}{\partial\epsilon}\Big|_{\epsilon=0}$$

The justification for the above expression for higher order terms, comes from the following identities[22]

$$\int_0^\infty \cos\omega_\alpha\tau\,\mathrm{d}\tau = 0 \qquad \int_0^\infty \sin\omega_\alpha\tau\,\mathrm{d}\tau = \frac{1}{\omega_\alpha} \qquad \int_0^\infty \tau^{2k}\cos\omega_\alpha\tau\,\mathrm{d}\tau = 0$$

and

$$\int_0^\infty \tau^{2k}\sin\omega_\alpha\tau\,\mathrm{d}\tau = \left(\frac{1}{\omega_\alpha}\right)^{2k+1}(2k!)$$

for $\omega_\alpha = \text{const} \neq 0$ and $k = \text{integer}$

To see the property of $\eta_{1\alpha}$, we write down the expression for $\eta_{1\alpha}$ more explicitly as

$$\begin{aligned}\eta_{1\alpha} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{2\omega_\alpha}\Bigg\{\sum_{\beta=1}^{n}\left(\frac{Q_\beta^0}{Q_\alpha^0}\right)\Big[r(\tau)P_{\alpha\beta}\Big(&\sin[(\omega_\alpha+\omega_\beta)\tau \\ &+\nu_\alpha^0+\nu_\beta^0] - \sin[(\omega_\alpha-\omega_\beta)\tau+\nu_\alpha^0-\nu_\beta^0]\Big) \\ &+C_{\alpha\beta}\omega_\beta\Big(\cos[(\omega_\alpha+\omega_\beta)\tau+\nu_\alpha^0+\nu_\beta^0] \\ &+\cos[(\omega_\alpha-\omega_\beta)\tau+\nu_\alpha^0-\nu_\beta^0]\Big)\Big]\Bigg\}\mathrm{d}\tau\end{aligned} \tag{17}$$

Since $r(t)$ is Gaussian, it is evident from Eq. (17) that $\eta_{1\alpha}$ is also Gaussian. Similarly, if we expand

$$\lim_{t\to\infty}\xi_\alpha = \xi_{1\alpha} + \epsilon\xi_{2\alpha} + 0(\epsilon^2)$$

we realize that $\xi_{1\alpha}$ is also Gaussian. However, $\eta_{2\alpha}, \xi_{2\alpha},\ldots$, are not Gaussian. Since $\eta_{1\alpha}$ and $\xi_{1\alpha}$ form the principal terms in the series expansions, we may regard $\lim_{t\to\infty}\epsilon\eta_\alpha$ and $\lim_{t\to\infty}\epsilon\xi_\alpha$ as almost Gaussian. As is shown in Appendix B, the departure from this assumption involves terms of order ϵ^3 and higher.

To discuss the stability in the mean square sense, we examine the behavior of the following quantities as t becomes large

$$\begin{aligned}E\{x_\alpha x_\beta\} &= Q_\alpha^0 Q_\beta^0 E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\sin(\omega_\alpha t+\nu_\alpha)\sin(\omega_\beta t+\nu_\beta)\right\} \\ &= \frac{1}{2}Q_\alpha^0 Q_\beta^0\Big[E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\cos[\epsilon(\xi_\alpha-\xi_\beta)t]\right\} \\ &\times\cos[(\omega_\alpha-\omega_\beta)t+\nu_\alpha^0-\nu_\beta^0] - E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\sin[\epsilon(\xi_\alpha-\xi_\beta)t]\right\} \\ &\times\sin[(\omega_\alpha-\omega_\beta)t+\nu_\alpha^0-\nu_\beta^0] - E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\cos[\epsilon(\xi_\alpha+\xi_\beta)t]\right\} \\ &\times\cos[(\omega_\alpha+\omega_\beta)t+\nu_\alpha^0+\nu_\beta^0] + E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\sin[\epsilon(\xi_\alpha+\xi_\beta)t]\right\} \\ &\times\sin[(\omega_\alpha+\omega_\beta)t+\nu_\alpha^0+\nu_\beta^0]\Big]\end{aligned} \tag{18}$$

Before going further, we introduce the following notation for convenience

$$\begin{aligned}&E\{\eta_\alpha\} = m_\alpha \qquad E\{(\eta_\alpha-m_\alpha)^2\} = \sigma_\alpha^2 \\ &E\{\xi_\alpha\} = M_\alpha \qquad E\{(\xi_\alpha-M_\alpha)^2\} = \mu_\alpha^2 \\ &E\{(\eta_\alpha-m_\alpha)(\eta_\beta-m_\beta)\} = \rho_{\alpha\beta}\sigma_\alpha\sigma_\beta \\ &E\{(\xi_\alpha-M_\alpha)(\xi_\beta-M_\beta)\} = \bar{\rho}_{\alpha\beta}\mu_\alpha\mu_\beta \\ &E\{(\eta_\alpha-m_\alpha)(\xi_\alpha-M_\alpha)\} = \gamma_{\alpha\alpha}\sigma_\alpha\mu_\alpha \\ &E\{(\eta_\alpha-m_\alpha)(\xi_\beta-M_\beta)\} = \gamma_{\alpha\beta}\sigma_\alpha\mu_\beta\end{aligned} \tag{19}$$

where $\rho_{\alpha\beta}$, $\bar{\rho}_{\alpha\beta}$, $\gamma_{\alpha\alpha}$, and $\gamma_{\alpha\beta}$ are the correlation coefficients. Using the properties of joint Gaussian distributions whose parameters are defined in Eqs. (19), we find that

$$E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\cos[(\xi_\alpha\mp\xi_\beta)\epsilon t]\right\} = e^{\pi_{1,2}}\cos(\Lambda_{1,2})$$

and

$$E\left\{e^{-\epsilon(\eta_\alpha+\eta_\beta)t}\sin[(\xi_\alpha\mp\xi_\beta)\epsilon t]\right\} = e^{\pi_{1,2}}\sin(\Lambda_{1,2})$$

where

$$\begin{aligned}\pi_{1,2} = -\epsilon t(m_\alpha+m_\beta) + \epsilon^2 t^2\Big[&\frac{1}{2}\left\{\left(\sigma_\alpha^2+\sigma_\beta^2\right) - \left(\mu_\alpha^2+\mu_\beta^2\right)\right\} \\ &+(\rho_{\alpha\beta}\sigma_\alpha\sigma_\beta \pm \bar{\rho}_{\alpha\beta}\mu_\alpha\mu_\beta)\Big]\end{aligned}$$

and

$$\begin{aligned}\Lambda_{1,2} = -\epsilon t(M_\alpha \mp M_\beta) + \epsilon^2 t^2(&\gamma_{\alpha\alpha}\sigma_\alpha\mu_\alpha \mp \gamma_{\beta\beta}\sigma_\beta\mu_\beta \\ &\mp\gamma_{\alpha\beta}\sigma_\alpha\mu_\beta + \gamma_{\beta\alpha}\sigma_\beta\mu_\alpha)\end{aligned}$$

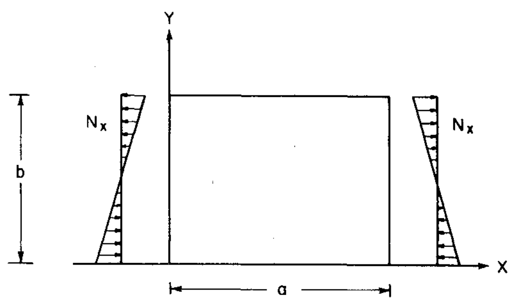


Fig. 3 A rectangular plate subject to stationary narrow-band Gaussian random forces.

In view of the above expressions and because of the fact that the correlation coefficients are bounded, i.e.,

$$-1 \leq \rho_{\alpha\beta} \leq 1 \qquad -1 \leq \bar{\rho}_{\alpha\beta} \leq 1$$

it is now evident that $E\{x_\alpha x_\beta\}$, $\alpha,\beta = 1,\ldots,n$, as given by Eq. (18), remain bounded as t tends to infinity, provided that

$$\lim_{t\to\infty} \left\{-(m_\alpha + m_\beta) + \frac{\epsilon t}{2}[(\sigma_\alpha + \sigma_\beta)^2 - (\mu_\alpha - \mu_\beta)^2]\right\} < 0$$

$$\alpha,\beta = 1,\ldots,n$$

Retaining only terms up to the order of ϵ, from Eqs. (16)-(18), we reduce this stability condition to

$$-\frac{1}{2}(C_{\alpha\alpha} + C_{\beta\beta}) + \lim_{t\to\infty}\left\{-\epsilon[\langle\eta_{2\alpha}\rangle + \langle\eta_{2\beta}\rangle] + \frac{\epsilon t}{2}\left[\left(\langle\eta_{1\alpha}^2\rangle^{1/2} + \langle\eta_{1\beta}^2\rangle^{1/2}\right)^2 - \left(\langle\xi_{1\alpha}^2\rangle^{1/2} - \langle\xi_{1\beta}^2\rangle^{1/2}\right)^2\right]\right\} < 0 \tag{20}$$

where $\langle\ldots\rangle$ stands for $E\{\ldots\}$.

Now let us define the autocorrelation function of $r(t)$ by $R(\tau_1 - \tau_2) = E\{r(\tau_1)r(\tau_2)\}$. This is related to the power spectrum, $S(\omega)$, by

$$R(\tau_1 - \tau_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega(\tau_1 - \tau_2)\,d\omega \tag{21}$$

Using Eqs. (17) and (21) and interchanging the order of integration, we obtain

$$\lim_{t\to\infty} t\langle\eta_{1\alpha}^2\rangle = \lim_{t\to\infty} t\langle\xi_{1\alpha}^2\rangle - \frac{1}{4\omega_\alpha^2}P_{\alpha\alpha}^2 S(0)$$

$$= \frac{1}{16\omega_\alpha^2}\int_{-\infty}^{\infty} S(\omega)\left\{P_{\alpha\alpha}^2[\delta(\omega + 2\omega_\alpha) + \delta(\omega - 2\omega_\alpha)] + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\left(\frac{Q_\beta^0}{Q_\alpha^0}\right)P_{\alpha\beta}^2\left[[\delta\left(\omega + (\omega_\alpha + \omega_\beta)\right) + \delta\left(\omega - (\omega_\alpha + \omega_\beta)\right)] - [\delta\left(\omega + (\omega_\alpha - \omega_\beta)\right) + \delta\left(\omega - (\omega_\alpha - \omega_\beta)\right)]\right]\right\}d\omega \tag{22}$$

where $\delta(\omega)$ is the Dirac delta function. Calculation for $\lim_{t\to\infty}\langle\eta_{2\alpha}\rangle$ is more involved algebraically. Nevertheless, the procedure is straightforward. We therefore give only the result, as follows

$$\lim_{t\to\infty}\langle\eta_{2\alpha}\rangle = \frac{1}{16}\int_{-\infty}^{\infty} S(\omega)\left\{\frac{P_{\alpha\alpha}^2}{\omega_\alpha}[\delta(\omega + 2\omega_\alpha) + \delta(\omega - 2\omega_\alpha)] + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{P_{\alpha\beta}P_{\beta\alpha}}{\omega_\alpha\omega_\beta}\left[[\delta\left(\omega + (\omega_\alpha + \omega_\beta)\right) + \delta\left(\omega - (\omega_\alpha + \omega_\beta)\right)] + [\delta\left(\omega + (\omega_\alpha - \omega_\beta)\right) + \delta\left(\omega - (\omega_\alpha - \omega_\beta)\right)]\right]\right\}d\omega \tag{23}$$

Also, $\lim_{t\to\infty}\langle\eta_{2\beta}\rangle$ has the same form as before (with interchanging the indices α and β).

Now, the frequencies ω_α, $\alpha = k,\ldots, n$, can be anywhere in the frequency domain. We shall discuss the combination resonance of the first kind in detail, and the other two cases can be treated in a similar manner. If the sum of any two natural frequencies, ω_α and ω_β, falls within the significant portion of the power spectrum, $S(\omega)$, i.e., if

$$\omega_c - \frac{\Delta\omega}{2} \leq \omega_\alpha + \omega_\beta \leq \omega_c + \frac{\Delta\omega}{2} \qquad \alpha\neq\beta \qquad \alpha,\beta = 1,\ldots,n$$

then we have

$$\lim_{t\to\infty} t\langle\eta_{1\alpha}^2\rangle = \lim_{t\to\infty} t\langle\xi_{1\alpha}^2\rangle = \frac{1}{8\omega_\alpha^2}\left(\frac{Q_\beta^0}{Q_\alpha^0}\right)^2 P_{\alpha\beta}^2 S(\omega_\alpha + \omega_\beta)$$

and

$$\lim_{t\to\infty}\langle\eta_{2\alpha}\rangle = \frac{1}{8\omega_\alpha\omega_\beta}P_{\alpha\beta}P_{\beta\alpha}S(\omega_\alpha + \omega_\beta)$$

Hence, the stability condition (20), becomes

$$-\frac{1}{2}(C_{\alpha\alpha} + C_{\beta\beta}) + \frac{\epsilon P_{\alpha\beta}P_{\beta\alpha}}{2\omega_\alpha\omega_\beta}S(\omega_\alpha + \omega_\beta) < 0$$

which can be rewritten as Eq. (4). Since $C_{\alpha\alpha}$ and $C_{\beta\beta}$ are positive and $S(\omega)$ is nonnegative for all ω, the stability condition is automatically satisfied when $P_{\alpha\beta}P_{\beta\alpha} < 0$, in which case the system remains stable without any additional conditions. Similarly, if the difference of any two natural frequencies, ω_α and ω_β, falls within the significant portion of the power spectrum $S(\omega)$, i.e., if $\omega_c - (\Delta\omega/2) \leq |\omega_\alpha - \omega_\beta| \leq \omega_c + (\Delta\omega/2)$, $\alpha\neq\beta$, $\alpha,\beta = 1,\ldots,n$, we obtain the stability condition, (5). And if $\omega_c - (\Delta\omega/2) \leq 2\omega_\alpha \leq \omega_c + (\Delta\omega/2)$, $\alpha = 1,\ldots, n$, we also arrive at the stability criterion (6).

V. Example

To illustrate our results, we consider a flat rectangular plate subjected to stationary narrow-band Gaussian random excitations. Structural damping is included, and the plate is assumed to be simply supported along its edges. Figure 3 shows the action of the forces. The linearized equation describing the bending motion of the plate is given by

$$\frac{\partial^4 w}{\partial X^4} + 2\frac{\partial^4 w}{\partial X^2\partial Y^2} + \frac{\partial^4 w}{\partial Y^4} = \frac{1}{D}\left(N_x\frac{\partial^2 w}{\partial X^2} - \epsilon\eta\frac{\partial w}{\partial t} - m\frac{\partial^2 w}{\partial t^2}\right) \tag{24}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity; h is the plate thickness; ν is Poisson's ratio; E is Young's modulus; w is the deflection of the plate in the z direction; m is the mass of the plate per unit area; $\epsilon\eta > 0$ denotes the structural damping; and N_x is the excitation force. N_x is distributed along the edges of the plate in the following way

$$N_x = [N_0 + \epsilon N_1 r(t)]\left(\frac{2Y}{b} - 1\right)$$

in which N_0 and N_1 are constants, b is the width of the plate, and $r(t)$ is a stationary narrow-band Gaussian random process with zero mean value and its power spectrum is denoted by $S(\omega)$. Again, the bandwidth $\Delta\omega$ of the significant portion of the power spectrum is assumed to be smaller than the absolute value of the difference between any two natural frequencies of the plate. Now let us set the solution form as

$$w(X,Y,t) = \sum_{\beta=1}^{2}\frac{2}{\sqrt{ab}}\sin\frac{\pi X}{a}\sin\frac{\beta\pi Y}{b}q_\beta(t) \tag{25}$$

which satisfies the boundary conditions. Substituting Eq. (25) into Eq. (24) and applying Galerkin's variational method, we obtain the following system of ordinary differential equations:

$$\ddot{q}_\alpha + \epsilon c\dot{q}_\alpha + \sum_{\beta=1}^{2}\left(\bar{A}_{\alpha\beta} + \epsilon\bar{P}_{\alpha\beta}r(t)\right)q_\beta = 0 \quad \alpha = 1,2 \tag{26}$$

where $c=\eta/m$

$$\bar{A}=[\bar{A}_{\alpha\beta}]=\begin{bmatrix}\Omega_1^2 & -\dfrac{32N_0}{9a^2m}\\ -\dfrac{32N_0}{9a^2m} & \Omega_2^2\end{bmatrix}$$

$$\bar{P}=[\bar{P}_{\alpha\beta}]=\begin{bmatrix}0 & -\dfrac{32N_t}{9a^2m}\\ -\dfrac{32N_t}{9a^2m} & 0\end{bmatrix}$$

in which $\Omega_\beta=\pi^2[(1/a^2)+(\beta^2/b^2)](D/m)^{1/2}$, $\beta=1,2$. Notice that both matrices $\bar{A}$ and $\bar{P}$ are symmetric. This is because the excitation force N_x is conservative. Equations (26) can be transformed to

$$\ddot{x}_\alpha+\epsilon c\dot{x}_\alpha+\sum_{\beta=1}^{2}\left(\omega_\beta^2+\epsilon P_{\alpha\beta}r(t)\right)x_\beta=0\quad \alpha=1,2 \tag{27}$$

where

$$x=\{x_\alpha\}=T^{-1}q\qquad T=\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}$$

$$P=[P_{\alpha\beta}]=\begin{bmatrix}u & v_1\\ -v_2 & u\end{bmatrix}$$

$$\lambda_{1,2}=\frac{9a^2m}{64N_0}\left\{\left(\Omega_1^2-\Omega_2^2\right)\mp\left[\left(\Omega_1^2-\Omega_2^2\right)^2+\left(\frac{64N_0}{9a^2m}\right)^2\right]^{1/2}\right\}$$

$$\omega_{1,2}^2=\frac{1}{2}\left\{\left(\Omega_1^2+\Omega_2^2\right)\pm\left[\left(\Omega_1^2-\Omega_2^2\right)^2+\left(\frac{64N_0}{9a^2m}\right)^2\right]^{1/2}\right\}$$

$$u=\frac{2N_tN_0(32/9a^2m)^2}{[(\Omega_1^2-\Omega_2^2)^2+(64N_0/9a^2m)^2]^{1/2}}$$

$$v_{1,2}=\frac{N_t(\Omega_1^2-\Omega_2^2)[(\Omega_1^2-\Omega_2^2)\pm\{(\Omega_1^2-\Omega_2^2)^2+(64N_0/9a^2m)^2\}^{1/2}]}{2N_0(32/9a^2m)^2\{(\Omega_1^2-\Omega_2^2)^2+(64N_0/9a^2m)^2\}^{1/2}}$$

Since $\Omega_2^2>\Omega_1^2$, it is easy to check that $v_1<0$, $v_2>0$. Consequently, we have $P_{12}P_{21}=v_1(-v_2)>0$. Equations (27) are linear differential equations with stationary narrow-band Gaussian random coefficients. Results presented in Sec. III can be directly applied here without any difficulties. Since $P_{12}P_{21}>0$, we have the following conclusions:

1) If $\omega_c-(\Delta\omega/2)\le(\omega_1+\omega_2)\le\omega_c+(\Delta\omega/2)$, then the condition under which the plate remains stable (in the mean square sense) is

$$\frac{\eta}{m}>\frac{\epsilon\,|v_1v_2|}{2\omega_1\omega_2}S(\omega_1+\omega_2)$$

2) If $\omega_c-(\Delta\omega/2)\le|\omega_2-\omega_1|\le\omega_c+(\Delta\omega/2)$, the plate will remain stable without any additional condition.

3) If $\omega_c-(\Delta\omega/2)\le 2\omega_\alpha\le\omega_c+(\Delta\omega/2)$, $\alpha=1,2$, then mean-square stability condition becomes

$$\frac{\eta}{m}>\frac{\epsilon}{2\omega_\alpha^2}u^2S(2\omega_\alpha)\qquad \alpha=1,2$$

Appendix A

In this appendix, we will compare our stability criteria in random excitation, i.e., conditions (4-6), with Ariaratnam's results.[9] Since three criteria are similar, we will compare combination resonance of the first kind here. Two other criteria can be treated in the same way.

In terms of our notation, Ariaratnam's criterion for combination resonance of the first kind is

$$\frac{4C_{\alpha\alpha}C_{\beta\beta}}{C_{\alpha\alpha}+C_{\beta\beta}}>\frac{\epsilon P_{\alpha\beta}P_{\beta\alpha}}{\omega_\alpha\omega_\beta}S(\omega_\alpha+\omega_\beta)$$

From the fact that $(C_{\alpha\alpha}-C_{\beta\beta})^2\ge 0$, we know that

$$(C_{\alpha\alpha}+C_{\beta\beta})^2\ge 4C_{\alpha\alpha}C_{\beta\beta} \tag{A1}$$

Since $C_{\alpha\alpha}$ and $C_{\beta\beta}$ are positive, Eq. (A1) is equivalent to

$$C_{\alpha\alpha}+C_{\beta\beta}\ge\frac{4C_{\alpha\alpha}C_{\beta\beta}}{C_{\alpha\alpha}+C_{\beta\beta}}$$

Hence we can see that the stability condition (4) is more conservative than Ariaratnam's result. However in applications the values of $C_{\alpha\alpha}$ and $C_{\beta\beta}$ are often small and very close to each other, i.e, $C_{\alpha\alpha}\approx C_{\beta\beta}$. Hence, the difference $|C_{\alpha\alpha}-C_{\beta\beta}|$ is very small so that $(C_{\alpha\alpha}-C_{\beta\beta})^2$ is of the order of ϵ^2 or even higher. This implies that, in most practical cases, $(C_{\alpha\alpha}-C_{\beta\beta})^2\approx 0$, which is equivalent to

$$C_{\alpha\alpha}+C_{\beta\beta}\approx\frac{4C_{\alpha\alpha}C_{\beta\beta}}{C_{\alpha\alpha}+C_{\beta\beta}}$$

Appendix B

Here we shall prove the following: If z_1 is a Gaussian random process, and z_2 is non-Gaussian, then for $0<\epsilon\ll 1$, $\epsilon z=\epsilon z_1+\epsilon^2z_2+0(\epsilon^3)$ is, up to the second order of approximation in ϵ, Gaussian.

To prove this statement, we let the characteristic function of ϵz be

$$\Phi_z(\omega)=E\{e^{\epsilon i\omega z}\}=E\{e^{\epsilon i\omega(z_1+\epsilon z_2+0(\epsilon^2))}\}\qquad i=\sqrt{-1}$$

which may be expressed as

$$\Phi_z(\omega)=E\Big\{1+i\omega\epsilon\left[z_1+\epsilon z_2+0(\epsilon^2)\right]+\frac{1}{2}\left[i\omega\epsilon\left(z_1+\epsilon z_2+0(\epsilon^2)\right)\right]^2+\ldots\Big\}$$

$$=1+\epsilon i\omega E\{z_1\}+\epsilon^2\left[\frac{1}{2}(i\omega)^2E\{z_1^2\}+i\omega E\{z_2\}\right]+0(\epsilon^3) \tag{B1}$$

On the other hand, we have

$$e^{i\omega\epsilon^2E\{z_2\}}E\{e^{\epsilon i\omega z_1}\}=1+\epsilon i\omega E\{z_1\}+\epsilon^2\left[\frac{1}{2}(i\omega)^2E\{z_1^2\}+i\omega E\{z_2\}\right]+0(\epsilon^3) \tag{B2}$$

Comparing Eqs. (B1) and (B2), we conclude that

$$\Phi_z(\omega)\approx e^{i\omega\epsilon^2E\{z_2\}}E\{e^{\epsilon i\omega z_1}\} \tag{B3}$$

Since z_1 is Gaussian with mean m_{z_1} and variance $\sigma_{z_1}^2$, we have

$$E\{e^{\epsilon i\omega z_1}\}=\exp\{im_{z_1}\epsilon\omega-\tfrac{1}{2}\sigma_{z_1}^2\epsilon^2\omega^2\} \tag{B4}$$

From Eqs. (B3) and (B4) we obtain

$$\Phi_z(\omega)\approx\exp\{i\omega(\epsilon m_{z_1}+\epsilon^2m_{z_2})-\tfrac{1}{2}\sigma_{z_1}^2\epsilon^2\omega^2\}$$

where $m_{z_2}=E\{z_2\}$. Thus, the random process ϵz is Gaussian with the following mean and variance

$$m_z=\epsilon m_{z_1}+\epsilon^2 m_{z_2} \qquad \sigma_z^2=\sigma_{z_1}^2\epsilon^2$$

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